Orthotropic Pressure Vessels with Axial Constraint

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The membrane stresses are determined which occur in two types of axisymmetric orthotropic pressure vessels with conditions of axial constraint for which the usual membrane analysis is inadequate The first type is the vessel with prescribed edge displacements The significant effect of orthotropy on the membrane stress distribution is demonstrated for circular, elliptical, and "zero-hoop-stress" meridians Some of the zero-hoop-stress meridians are shown to be well-suited for a shell with a light skin and heavy meridional ribs or filaments The second type of vessel considered is the toroid with arbitrary asymmetric cross section In addition to the membrane stresses, the high-bending stresses due to the asymmetry of the cross section are determined

Nomenclature

$r\ z$	= cylindrical coordinates of the midsurface of a shell of revolution
φ	= angle between normal to midsurface and axis
r_1	= radius of curvature of meridian
r_2	= $r/\sin\varphi$, circumferential radius of curvature
λ	= r_2/r_1
N_φ, N_θ	= meridional and circumferential stress resultants
w, v	= normal and tangential components of midsurface displacement
$\delta\ \delta$	= radial and axial components of midsurface displacement
$\epsilon_\varphi, \epsilon_\theta$	= meridional and circumferential strains
h	= thickness
C_{11}, C_{12}, C_{22}	= moduli of elasticity of orthotropic material
Δ	= $C_{11}C_{22} - C_{12}^2$

Introduction

PRESSURE vessels have received such a portion of the literature of shell theory that any additional work on the subject seems superfluous, particularly if attention is restricted to the problem of determining the membrane stress distribution in vessels of revolution

However, practical design configurations can occur in which the pressure vessel is subjected to conditions of axial constraint For such problems, the axial load is indeterminate from equilibrium considerations, and so the usual analysis, based on only equilibrium considerations, is inadequate In this paper, a few properties of the "edge-effect" or "bending" solutions are used to obtain the membrane stresses in two types of pressure vessels for which the axial load is statically indeterminate The first type consists of a thin shell with both edges attached to a rigid core After consideration of such possibilities as relative thermal expansion, the analysis of the first type of vessel is seen to be equivalent to the problem of the analysis of a shell of revolution with arbitrarily prescribed displacement edge conditions The second type of pressure vessel discussed in this paper is the toroid of arbitrary asymmetric cross section The requirement that stresses and deflections must be continuous around the meridian is the condition of axial constraint The asymmetry of the cross section causes high-bending stresses; however, for some design problems, the advantage of an asymmetric shape may outweigh the disadvantage of the increased stresses

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General Solution

Membrane Solution

The axisymmetric membrane solution for a shell of revolution is given on p 23 of Ref 1 For constant pressure loading, the membrane stress resultants are

$$N_\varphi = (pr_2/2) + (V/2\pi r \sin\varphi) \tag{1a}$$

$$N_\theta = (pr_2/2)(2 - \lambda) - (V/2\pi r_1 \sin^2\varphi) \tag{1b}$$

where p is the pressure and V is an arbitrary constant

Since Eqs (1a) and (1b) are obtained from only equilibrium considerations, it is easily shown that the constant V is the resultant axial force acting at the apex (where $r = 0$) of a dome-shaped shell [1] Therefore, for the usual simply connected closed pressure vessel, $V = 0$ Equations (1a) and (1b) also provide a solution for toroids At the points of the toroid meridian where $\varphi = 0$ and π, the radius of curvature r_2 is infinite Thus, in order for the membrane stresses given by Eqs (1a) and (1b) to be finite and continuous, it must be that $r(0) = r(\pi)$ and that $V = -\pi p r^2(0)$, as discussed on p 33 of Ref 1

Since the constant V is easily determined for the simply-connected vessel and the toroid with symmetric cross section, which are the configurations usually encountered, the problem of over-all stress analysis usually requires nothing in addition to Eqs (1a) and (1b) (Discontinuities in geometry, radial constraints, etc , produce only local deviations of the stress distribution [1]) In this paper, the primary concern is for vessels for which the computation of V requires more than simple equilibrium considerations For vessels with prescribed edge displacements, it is necessary for the computation of V to know the displacements produced by the membrane stress resultants, Eqs (1a) and (1b)

For an orthotropic shell of revolution with the lines of elastic symmetry coinciding with meridians and parallels, the relations between stress and strain are given by

$$N_\varphi = h(C_{11}\epsilon_\varphi + C_{12}\epsilon_\theta) \tag{2a}$$

$$N_\theta = h(C_{12}\epsilon_\varphi + C_{22}\epsilon_\theta) \tag{2b}$$

The displacement-strain relations are given on p 98 of Ref 1

$$v = \sin\varphi \left(\int_{\varphi_c}^{\varphi_d} \frac{q}{\sin\varphi}\, d\varphi + \frac{v_c}{\sin\varphi} \right) \tag{3a}$$

$$w = r_2\epsilon_\theta - v\cot\varphi \tag{3b}$$

where v is the value of v at $\varphi = \varphi$ and where

$$q = r_1\epsilon_\varphi - r_2\epsilon_\theta$$

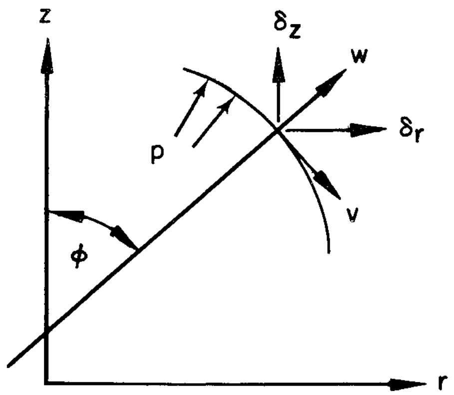

Fig 1 Meridian curve

After the inversion of Eqs (2a) and (2b), and the substitution of the stress resultants Eqs (1a) and (1b), one obtains for q the result

$$q = \tfrac{1}{2}pq_p + (1/2\pi)Vq_V \tag{3c}$$

where

$$q_p = \frac{C_{22} + 2(\lambda - 1)C_{12} + \lambda(\lambda - 2)C_{11}}{h\Delta} r_1r_2$$

$$q_V = \frac{C_{22} + 2\lambda C_{12} + \lambda^2C_{11}}{h\Delta r^2} r_1r_2$$

The components of deflection in the radial and axial directions (see Fig 1a) are

$$\delta_r = r\epsilon_\theta = w\cos\varphi + v\sin\varphi = \frac{p}{2}\frac{rr_2}{h\Delta}[(2 - \lambda)C_{11} - C_{12}] - \frac{V}{2\pi r^2}\frac{rr_2}{h\Delta}[\lambda C_{11} + C_{12}] \tag{3d}$$

$$\delta_z = v/\sin\varphi + \delta \cot\varphi \tag{3e}$$

The preceding results are valid for the orthotoropic shell of variable thickness and variable elastic moduli Naturally, orthotropic shell analysis has its greatest utility in the investigation of composite structures, which are orthotropic only in a macroscopic sense In Ref 2, the equivalent orthotropic moduli of elasticity are obtained for shells consisting of layers of filaments embedded in a matrix For a shell consisting of an isotropic skin with meridional and circumferential stiffeners attached, the formulas for the equivalent moduli of elasticity are

$$C_{11} = E/(1 - \nu^2) + E_\varphi h_\varphi/h \tag{4a}$$

$$C_{12} = \nu E/(1 - \nu^2) \tag{4b}$$

$$C_{22} = E/(1 - \nu^2) + E_\theta h_\theta/h \tag{4c}$$

where E, ν, and h are the moduli and thickness of the skin, E_φ and h_φ are the modulus and average thickness (area of cross section divided by spacing) of the meridional stiffeners, and E_θ and h_θ are the modulus and average thickness of the circumferential stiffeners The (membrane) stresses in the skin are

$$\sigma_\varphi = \frac{E}{h\Delta(1 - \nu^2)}[(C_{22} - \nu C_{12})N_\varphi + (\nu C_{11} - C_{12})N_\theta] \tag{5a}$$

$$\sigma_\theta = \frac{E}{h\Delta(1 - \nu^2)}[(\nu C_{22} - C_{12})N_\varphi + (C_{11} - \nu C_{12})N_\theta] \tag{5b}$$

whereas the (membrane) stresses in the stiffeners are

$$\sigma_\varphi = \frac{E_\varphi}{h\Delta}[C_{22}N_\varphi - C_{12}N_\theta] \tag{5c}$$

$$\sigma_\theta = \frac{E_\theta}{h\Delta}[-C_{12}N_\varphi + C_{11}N_\theta] \tag{5d}$$

Bending Solution for Dome

The asymptotic (as $h/r \to 0$) form of the particular solution to the equations of the general bending theory is given in Ref 3 This approximate solution is uniformly valid in the "steep" and "shallow" regions of the homogeneous, orthotropic, dome-shaped shell This solution is obtained in terms of Bessel and Lommel's functions, but is equivalent to the preceding membrane solutions in the steep region, the region where

$$\rho|\sin\varphi| \gtrsim 10 \tag{6}$$

where ρ is large when the shell is thin

$$\rho = (12\Delta r_1^4/C_{11}^2r_2^2h^2)^{1/4}$$

For nonhomogeneous shells, ρ will be modified but will remain $0[(r/h)^{1/2}]$

Bending Solution for Toroidal Segment

The criterion equation (6) for the validity of the membrane solution is somewhat too restrictive for toroidal shells For a toroidal segment whose meridian has only one point at which $\sin\varphi = 0$, as shown in Fig 2, the arbitrary constant of Eq (1) is most conveniently written as

$$V = V_j - \pi pe_j^2 \tag{7}$$

where V_j is a new arbitrary constant, and e_j is the radial distance to the point at which $\sin\varphi = 0$ Equation (1) is now

$$N_\varphi = \frac{pr_2}{2}\left(1 - \frac{e_j^2}{r^2}\right) + \frac{V_j}{2\pi r \sin\varphi} \tag{8a}$$

$$N_\theta = \frac{pr_2}{2}\left(2 - \lambda + \lambda\frac{e_j^2}{r^2}\right) - \frac{V_j}{2\pi r_1 \sin^2\varphi} \tag{8b}$$

The terms of Eq (8), multiplied by the pressure p, are finite at $\sin\varphi = 0$, whereas the terms multiplied by V_j are singular The regular terms of Eq (8), when substituted in Eq (3), produce displacements that are singular at $\sin\varphi = 0$ [1] However, a small amount of bending actually takes place which smooths the displacements with only negligible $0[(h/r)^{1/3}]$ additional bending stress [4 5] On the other hand, substantial bending effects are caused by the axial load V_j The following results were obtained in Ref 6 for the toroid of circular cross section and generalized in Ref 4 to the general meridian with a finite, nonzero radius of curvature at $\sin\varphi = 0$ The stress resultants near $\sin\varphi = 0$, valid when $\mu_j^{1/3} \gtrsim 10$, are

$$N_\varphi = \mu_j^{1/3}T_i(x_j)V_j/2\pi r \tag{9a}$$

$$N_\theta = \mu_j^{2/3}T_i'(x_j)V_j/2\pi r_1 \tag{9b}$$

$$6M_\varphi/h = (3C_{11}^2/\Delta)^{1/2}\mu_j^{2/3}T\,'(x_j)V_j/2\pi r_1 \tag{9c}$$

$$M_\theta = M_\varphi C_{12}/C_{11} \tag{9d}$$

where

$$\mu_j = [(12\Delta/C_{11}^2)^{1/2}r_1^2/rh]_{=j}$$

$$x_j = \mu_j^{1/3}\sin\varphi$$

and where T_r and T_i are functions tabulated in Ref 6 with the behavior for $|x| \gtrsim 5$

$$T_i(x) \sim 1/x$$

$$T_r(x) \sim 2/x^4$$

and with the extreme values

$$T_i(\pm 1\ 5) = \pm 0\ 846$$
$$T_i'(0) = 0\ 939$$
$$T\,'(\pm 1\ 22) = \pm 0\ 753$$

Thus, the direct stresses, Eqs (9a) and (9b), become equivalent to the V_j contribution of Eqs (8a) and (8b), and the bending stresses, Eqs (9c) and (9d), become negligible at a distance from the apex given by $|x_j| \gtrsim 5$, which agrees with the criterion equation (6)

Thus, for the very thin toroid, the deviation from the membrane stress distribution, Eqs (8a) and (8b), is restricted to a region near $\sin\varphi = 0$ In contrast, the over-all deformation of the toroid is quite dependent upon the bending behavior The regular membrane solution, the terms of Eq (8) multiplied by p, produces deflections of the order of magnitude

$$\delta = 0(pr^2/C_{11}h)$$

However, the axial load V_j produces the relative axial displacement between any two points of the meridian φ and φ_d[4, 6]

$$\delta_z(\varphi_c) - \delta(\varphi_d) = b_j V_j \pi^{-1} \int_{x_j(\varphi_d)}^{x_j(\varphi_c)} T_r(x)dx$$

where

$$b_j = [(3/\Delta)^{1/2} r_1/h^2]_{=j} \tag{9e}$$

Since the integral is rapidly convergent to its limiting value

$$\int_{-2}^{2} T_r(x)dx \approx \int_{-\infty}^{\infty} T(x)dx \equiv -\pi$$

the realtive displacement between any point on the outer region of the toroidal segment $\sin\varphi_c \gtrsim 2\mu_j^{-1/3}$ and any point on the inner region $\sin\varphi_d \gtrsim -2\mu_j^{-1/3}$ is given by the constant

$$\delta_j = b_j V_j \tag{9f}$$

The radial displacement due to V_j is negligible Thus, the axial load V_j causes the simple deformation of the toroidal segment indicated by the dashed curve in Fig 2 The inner and outer regions undergo a relative rigid body axial displacement of magnitude δ_j; the complications of the deformation state all occur in the region near the apex Note that, if $V_j = 0(pr^2)$, then the deformation due to the regular membrane solution is $0(h/r)$ and is thus negligible in comparison with the relative rigid body displacement given by Eq (9f)

Edge-Effect Solutions

In addition to the particular solution, because of axial and surface loading, complementary solutions to the equations of the general bending theory exist that are often referred to as "edge-effect" solutions because of the high rate at which their magnitude diminishes with the meridional distance Asymptotic (as $h/r \to 0$) forms of the complementary solutions are obtained in Ref 3 that are uniformly valid in the steep and shallow regions of a homogeneous orthotropic shell Only these properties of the complementary solutions are required for the present investigation: 1) because of the "edge-effect" behavior these solutions generally have little influence on the over-all design of shells; 2) the complementary solutions give a stress system which has zero axial resultant force; and 3) for $\rho|\varphi| \gtrsim 10$, the complementary solutions give a tangential displacement that is of a lower order of magnitude than the normal displacement, i e , $v = 0(\rho^{-1}w)$

Boundary Conditions

The general solution is the sum of the membrane (with the necessary bending corrections) and edge-effect solutions The four constants of the edge-effect solution and the constant V of the membrane solution are determined from five linearly independent relations that must be prescribed between the edge values of radial force, radial displacement, moment, rotation, axial load at one edge, and the difference in axial displacement at the two edges However, there are two important cases for which the general equations uncouple, so that the constant V of the membrane solution can be determined independently from a precise knowledge of the edge-effect solution

Axial Load Prescribed

The edge-effect solution gives a system of stresses that have zero axial resultant Hence, if the resultant axial force is prescribed at one edge of the shell, the constant V is easily determined This is the situation commonly encountered and so quickly assumed by most authors that it may seem to be the only possibility

Tangential Displacement Prescribed

The edge-effect solution gives a tangential displacement that is negligibly small in comparison with the normal displacement Hence, the tangential displacement of the membrane solution alone must equal the prescribed values at both edges Thus, directly from Eq (3), one obtains the expressions for the constant V:

$$(V/2\pi)I_3 = (p/2)I_1 + I_2 \tag{10}$$

where

$$I_1 = \int_{\varphi}^{\varphi_d} \frac{q_p}{\sin\varphi} d\varphi$$

$$I_2 = \left(\frac{v}{\sin\varphi}\right)_{\varphi_d} - \left(\frac{v}{\sin\varphi}\right)_{\varphi} =$$

$$\delta(\varphi_d)\cot\varphi_d - \delta(\varphi)\cot\varphi + \delta(\varphi) - \delta(\varphi_d)$$

$$I_3 = \int_{\varphi}^{\varphi_d} \frac{q_V}{\sin\varphi} d\varphi$$

Unlike the preceding case for prescribed axial load, this result for V is approximate, but will be valid if the stresses of the edge-effect solution are of a magnitude less than or equal to the membrane stresses That is, Eq (10) will be valid unless the edge normal displacement and/or rotation are prescribed to be much larger than the tangential displacement

Mixed Conditions

Since the tangential displacement vector does not generally coincide with the axial stress resultant vector, mixed boundary conditions of elastic constraint cannot be handled by the membrane solution alone

Shells with Prescribed Edge Displacement

The membrane stresses [Eq (1)], when the displacements are prescribed at both edges of the shell, may be more con-

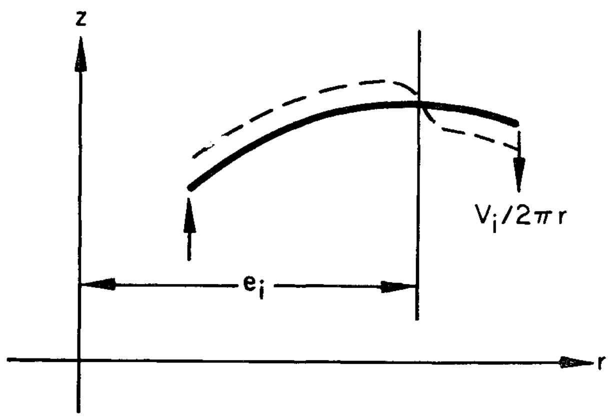

Fig 2 Meridian of toroid segment with axial load

veniently written in the form

$$2N_\varphi/r_2 = p + \alpha(r_c/r)^2(p - p_0) \tag{11a}$$

$$2N_\theta/r_2 = p(2 - \lambda) - \alpha\lambda(r_c/r)^2(p - p_0) \tag{11b}$$

where, from Eq (10),

$$\alpha = (I_1/r_c^2 I_3) \qquad p_0 = -(2I_2/I_1) \tag{11c}$$

and where $r_c = r(\varphi_c)$ For a shell with both edges attached to a rigid base, the quantity p_0 is a measure of the mismatch between shell and base When $p = p_0$ or $\alpha = 0$, the stresses are identical with the membrane stresses in a closed shell with no constraints However, in general, $\alpha = 0(1)$ and $p \neq p_0$ so that the constraints at the shell edges do have a significant effect There is little difficulty in computing α numerically for arbitrary variations in orthotropy, thickness, and meridian shape, but the behavior of the parameter α may be seen more clearly by considering constant orthotropic properties and simple meridians for which the integrations may be accomplished analytically For further simplification, the meridians considered will be symmetric about $\varphi = \pi/2$ so that $\varphi_d = \pi - \varphi$, as shown in Fig 3

Sphere

For the circular meridian $\lambda = 1$ the parameter α is given by

$$\alpha = \left[\frac{2(C_{11} - C_{22}) \log \cot\varphi/2}{(C_{11} + 2C_{12} + C_{22})(\cos\varphi + \sin^2\varphi \log \cot\varphi/2)}\right]_{\varphi=\varphi} \tag{12}$$

If the shell has the same stiffness in meridional and circumferential directions, $C_{22}/C_{11} = 1$ and $\alpha = 0$, and so the edge constraints cause no additional membrane stress However, for general orthotropy, α will become large as $\varphi \to 0$

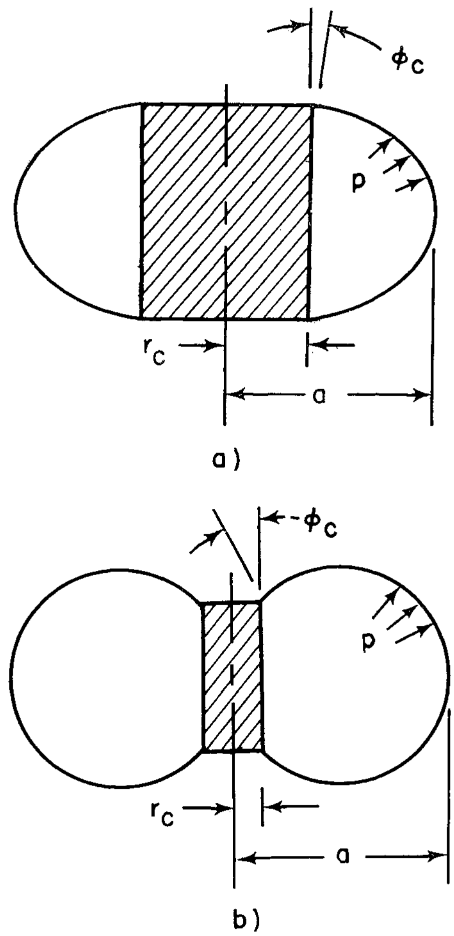

Fig 3 Pressure vessels with edges attached to rigid cylinder

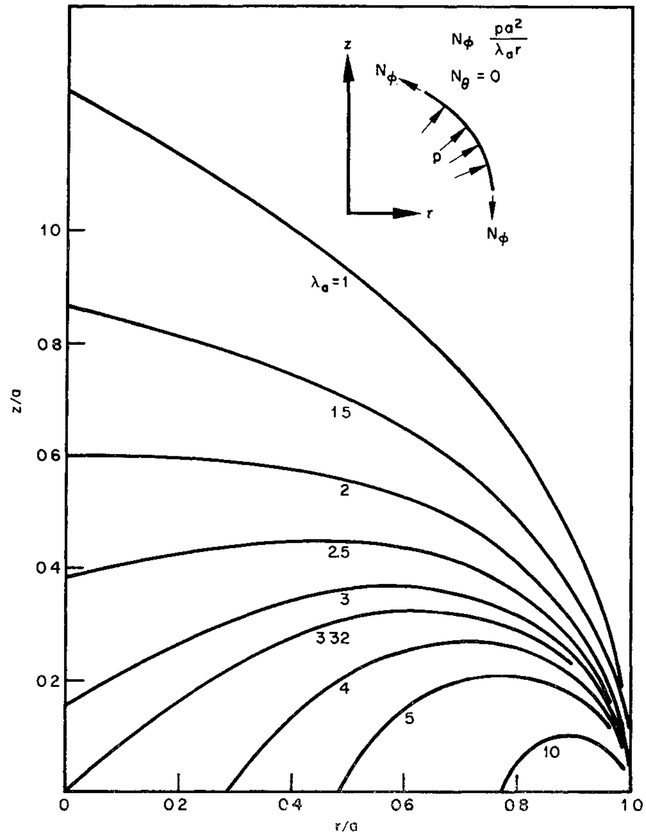

Fig 4 Zeıo-hoop-stress meridians

Ellipsoid

For the elliptical meridian

$$\lambda = 1 + k \sin^2\varphi \qquad br_1/a^2 = \lambda^{-3/2}$$

$$k = (a^2/b^2) - 1$$

where $r(\pi/2) = a$, one obtains for $k \geq 0$

$$\alpha = 2[\lambda\beta/\gamma]_{\varphi=\varphi} \tag{13}$$

where

$$\beta = (C_{11} - C_{22}) \log \cot\left(\frac{\varphi}{2}\right) + (C_{22} - 2C_{12}) \frac{k}{2(1+k)} \frac{\cos\varphi}{\lambda} + \left[\frac{C_{22} - 2C_{12}}{4(1+k)} + \frac{1}{2} C_{22} - C_{11}\right] \times \left(\frac{k}{1+k}\right)^{1/2} \log\frac{1 + [k/(1+k)]^{1/2}\cos\varphi}{1 - [(k/1+k)]^{1/2}\cos\varphi}$$

$$\gamma = (C_{11} + 2C_{12} + C_{22})[\cos\varphi + \sin^2\varphi \log \cot(\varphi/2)] + (C_{11} - C_{22})2k \sin^2\varphi \log \cot(\varphi/2) + kC_{22}\left(\frac{k}{1+k}\right)^{1/2} \sin^2\varphi \log\frac{1 + [k/(1+k)]^{1/2}\cos\varphi}{1 - [k/(1+k)]^{1/2}\cos\varphi}$$

For $k \to 0$, the expression (13) becomes equal to Eq (12) which is positive for $C_{22}/C_{11} < 1$, i e , for shells stiffer in the meridional direction As k becomes large, $\alpha \to -1$, hence, α is equal to zero for some value of k when $C_{22}/C_{11} < 1$ Indeed, for $C_{22}/C_{11} < 1$ and $\varphi \ll 1$, one has $\alpha = 0$ for

$$\varphi_c \approx \exp\left[-\frac{2C_{11} - C_{22}}{2(C_{11} - C_{22})} \log 2k\right]$$

However, it may be advantageous to select geometry that gives a value of α that optimizes the total stress distribution, Eqs (11a) and (11b), in some sense For instance, the requirement that the circumferential stress be tensile (from buckling considerations) for the vessel with external pressure, i e , $2N_\theta/pr_2 < 0$ is satisfied for $p_0 = 0$ if $\alpha \gtrsim -1 + 2/\lambda(\varphi)$ Generally, the meridional and hoop-stress resultants will be of the same order of magnitude for any orthotropy

Zero-Hoop-Stress Shell

A desirable shell structure, from several viewpoints, consists of a light skin with meridional ribs or filaments that carry most of the load Such a construction used with an elliptic meridian will generally be inefficient, since the hoop and meridional loads remain of the same magnitude for any orthotropy However, there does exist a class of shells for which N_θ is identically zero (with the proper boundary conditions) which is ideal for the meridional rib-light skin construction

Setting Eq (1b) equal to zero gives the equation

$$2 - \lambda - C_1\lambda/r^2 = 0$$

where C_1 is a constant Since

$$r_1 \cos\varphi = dr/d\varphi \qquad \lambda = r/r_1 \sin\varphi$$

the equation may be written as

$$(dr^2/d\varphi) \tan\varphi - r^2 = C_1$$

which is a first-order linear equation with the solution

$$r^2 = -C_1 + C_2 \sin\varphi$$

The two constants are determined from the condition that at $\varphi = \pi/2, r = a$, and $\lambda = \lambda_a$ The result is that the meridian curves which satisfy the zero-hoop-stress condition may be described by

$$(r/a)^2 = \lambda \sin\varphi/\lambda_a \tag{14a}$$

$$rr_1 = a^2/\lambda_a \tag{14b}$$

where

$$\lambda = 2 + (\lambda_a - 2)/\sin\varphi$$

and

$$\frac{z}{a} = \int \frac{r_1}{a} \sin\varphi\, d\varphi = \lambda_a^{-1/2} \int_\varphi^{\pi/2} \frac{\sin\varphi\, d\varphi}{(\lambda_a - 2 + 2\sin\varphi)^{1/2}} = \lambda_a^{-1/2}\left[2\int_0^\psi (1 - k^2 \sin^2\psi)^{1/2} d\psi - \int_0^\psi \frac{d\psi}{(1 - k^2\sin^2\psi)^{1/2}}\right] \tag{14c}$$

where

$$\psi = \cos^{-1} r/a \qquad k^2 = \lambda_a/4$$

The membrane stress resultants are

$$N_\varphi = (pa^2/\lambda_a r) \qquad N_\theta = 0 \tag{15}$$

Since rN_φ is constant, the force in the meridional ribs will also be constant (if all the load is carried by the ribs) Hence, the zero-hoop-stress shells are the members of the general class of "isotensoid" shells discussed in Ref 7 for which the filaments coincide with meridians An interesting analogy is given in Ref 7 between the meridian curve of the zero-hoop-stress shell and the deformed shape of the eccentrically loaded column

The meridian curves, Eq (14), are shown in Fig 4 The curve for $\lambda_a = 4$ is asymptotic to the z axis For $\lambda_a > 4$, the curves form "regressive" loops[7] with $\varphi = -\pi/2$ at $(r/a)^2 = 1 - 4/\lambda_a$ The curve for $\lambda_a = 2$ is the only one which gives a closed vessel free from discontinuities in the slope of the meridian, and thus free from the need of an

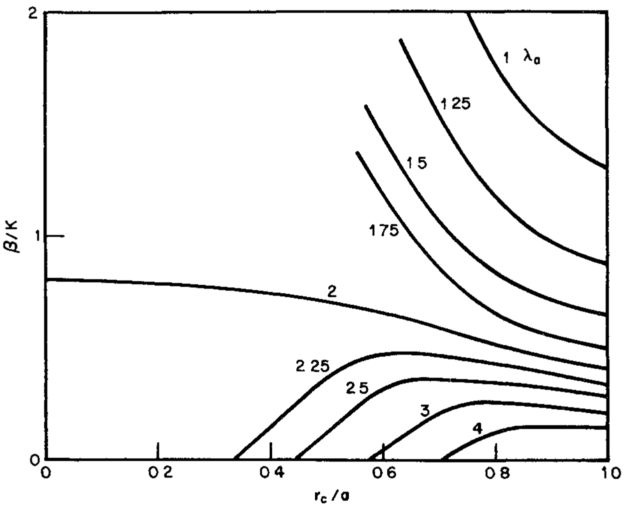

Fig 5 Factor β/K for shell clamped at $r = r$

external force to maintain the zero-hoop-stress condition A plausible means of providing the external force would be to add an axial rod, which, for internal pressure, would be in compression for $\lambda_a < 2$ and in tension for $2 < \lambda_a < 3\ 32$, or a relatively inextensional ring for $\lambda_a > 3\ 32$ However, the actual effects of the (rigid) edge constraint will be obtained from Eqs (10) and (11)

If a new constant β is introduced

$$\alpha = [(2/\lambda_a) - 1](a^2/r_c^2) - (2/\lambda_c)\beta$$

then the stresses, Eq (11), for the zero-hoop-stress meridians [Eq (14)] may be written as

$$N_\varphi/pr_1 = 1 - \beta(\sin\varphi_c/\sin\varphi) \tag{16a}$$

$$N_\theta/pr_2 = \beta(\sin\varphi_c/\sin\varphi) \tag{16b}$$

The constant β thus gives the magnitude of the deviation of the actual stress distribution from the "ideal" distribution, Eq (15) From the general formula for α [Eq (11c)] the expression for β is obtained:

$$\beta = \frac{a^2}{r^2}\frac{\displaystyle\int_\varphi^{\pi/2} \frac{C_{22} + \lambda C_{12}}{\lambda h\Delta \sin^2\varphi}\, d\varphi}{\displaystyle\int_{\varphi_c}^{\pi/2} \frac{C_{22} + 2\lambda C_{12} + \lambda^2 C_{11}}{\lambda h \Delta \sin^3\varphi}\, d\varphi} \tag{17}$$

It is evident that β will become small as the shell is made stiffer in the meridional direction, i e , as C_{11} becomes large Consider a shell with only meridional stiffeners and with constant orthotropic properties and with

$$K = Eh/E_\varphi h_\varphi \ll 1$$

Then Eq (17) is approximately, for $\lambda^2 \gg K$,

$$\frac{\beta}{K} = \frac{a^2}{r^2}\frac{\displaystyle\int_\varphi^{\pi/2} \frac{1 + \nu\lambda}{\lambda \sin^2\varphi}\, d\varphi}{\displaystyle\int_\varphi^{\pi/2} \frac{\lambda}{\sin^3\varphi}\, d\varphi} \tag{18}$$

The closed-form expression for the integrals can be obtained; of more interest are the curves in Fig 5 giving the behavior of Eq (18) for various values of λ_a and for $\nu = 0\ 3$ For the behavior at the limits, one obtains

$$\beta/K \to (1 + \nu\lambda_a)/\lambda_a^2 \quad \text{as} \quad \varphi_c \to \pi/2$$

and for $\lambda_a > 2$,

$$\beta/K = 0 \quad \text{for} \quad \varphi_c \leq 0$$

For $\lambda_a < 2$, β/K becomes large as $r \to 0$, and the approximate form Eq (18) becomes invalid since $\lambda_c \to 0$; however, Eq (17) gives

$$\beta \to (a^2/r_c^2)\sin\varphi \to (a^2/r_c^2)[1 - (\lambda_a/2)] \quad \text{as} \quad r_c/a \to 0$$

The curves of Fig 5 clearly show the strong dependence of β on the choice of the meridian For $\lambda_a = 2$, β varies smoothly between 0 4K and 0 8K, and so, despite the edge constraints, the meridional stiffeners will carry their full share of the load For λ_a even slightly greater than 2, the value is significantly reduced so that the meridional stiffeners carry even more of the load However, for λ_a slightly less than 2, the value of β may be quite large In Ref 7, the profiles for $\lambda_a < 2$ are rejected on the grounds that the axial rod will be in compression when the vessel is subjected to internal pressure; now they may be rejected on the more serious grounds of being much too sensitive to the conditions of edge constraint for internal or external pressure

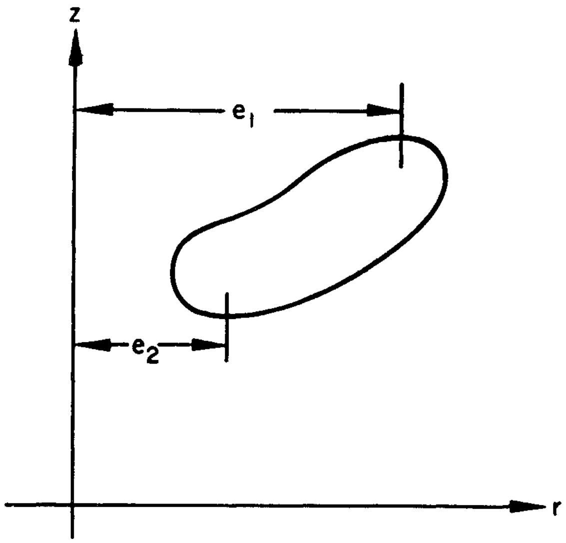

Fig 6a Meridian of toroid with asymmetric cross section

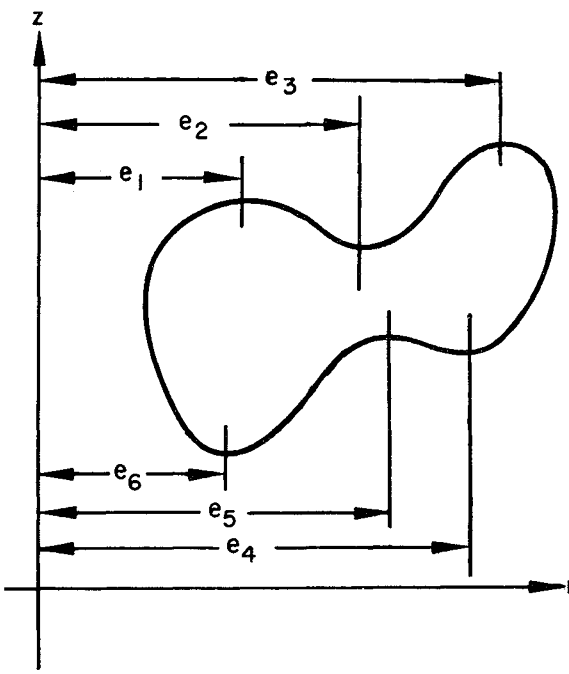

Fig 6b Toroid meridian with six points of horizontal tangency

The validity of the pleasant behavior of the shells with meridians $\lambda_a > 2$ may be questioned because the membrane solutions are generally invalid in the neighborhood of $\varphi = 0$ However, the result from the bending theory, Eq (9f), shows that a small axial load acting on a toroidal shell segment, as shown in Fig 2, causes a large relative axial deflection at the two edges Hence, the deflections of the membrane solution at the edges of such vessels, as shown in Fig 3b, are made compatible with the constraint conditions by quite small additional axial forces In other words, a complete bending analysis would only give the result that the curves of Fig 5 for $\lambda_a > 2$ would approach $0(h/r)$ instead of zero for $\varphi \leq 0$

The continuation of the curve of Fig 5 for $\lambda_a = 2$ to $r_c/a = 0$ may be misleading, since the analysis is only valid for $\varphi \gtrsim 10\rho^{-1}$ The justification is that the shell may be made arbitrarily thin so that $\rho \to \infty$, and allowable r/a may become arbitrarily small

Toroids with Asymmetric Cross Section

The determination of the unknown axial load V of Eq (1) for a toroidal pressure vessel with asymmetric cross section, such as shown in Fig 6a, depends primarily on the result given by Eq (9f) First, the two axial loads V_1 and V_2 are introduced:

$$V = V_1 - \pi p e_1^2 \tag{19a}$$

$$V = V_2 - \pi p e_2^2 \tag{19b}$$

where e_1 and e_2 are the radial distances to the points at which $\sin\varphi = 0$

The relative vertical deflection of the inner and outer portion of the shell near $r = e_1$ is obtained from Eqs (9e) and (9f):

$$\delta_1 = b_1 V_1 \tag{20a}$$

whereas the relative vertical deflection of the inner and outer portions near $r = e_2$ is

$$\delta_2 = b_2 V_2 \tag{20b}$$

The condition of continuity of displacement is satisfied if

$$\delta_1 + \delta_2 = 0 \tag{21}$$

Equations (19–21) yield this result for the unknown axial load:

$$V = -\pi p(b_1 e_1^2 + b_2 e_2^2)/(b_1 + b_2) \tag{22}$$

Thus, the computation of V is very simple and requires only a knowledge of the shell properties at the two points at which $\sin\varphi = 0$ With V known, the over-all stress distribution is easily computed from Eq (1) The details of the stress concentrations near $r = e_1$ are given by Eqs (9a–9d) with $j = 1$ and V_1 obtained from Eq (19a) Similarly, the stresses near the point of horizontal tangency $r = e_2$ are given by Eqs (9a–9d) with $j = 2$ and V_2 obtained from Eq (19b)

Note that when $e_1 = e_2$, the result is

$$V_1 = V_2 = 0$$

and the bending stress concentrations do not occur Thus, from consideration of stresses alone, it is desirable to have $e_1 = e_2$

The preceding analysis for the cross section with two points at which $\sin\varphi = 0$ is easily generalized to a cross section with n such points For example, the cross section shown in Fig 6b has $n = 6$ A set of axial loads V_j are introduced which satisfy

$$V = V_j - \pi p e_j^2 \quad \text{for} \quad j = 1,2, \quad , n \tag{23}$$

The deformation near the point $r = e_j$ is given by

$$\delta_j = |b_j| V_j \tag{24}$$

where b_j is defined by Eq (9e) Continuity of the deformation requires that

$$\sum_{j=1\ 2}^{n} \delta_j = 0 \tag{25}$$

The solution of Eqs (23–25) is

$$V = -\pi p \sum_{j=1}^{n} |b_j| e_j^2 \Big/ \sum_{j=1}^{n} |b_j| \tag{26}$$

As before, V, which gives the membrane stresses [Eq (1)] is easily computed The quantities V_j, which give the bending stresses [Eqs (9a–9d)] at each point $r = e_j$, are easily obtained from Eq (23)

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Large-Amplitude Vibration and Response of Curved Panels

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The dynamic nonlinear shallow-shell equations are examined in the special case of a cylindrical shell segment Different developments are given for two systems: in system A, the stress boundary conditions are satisfied exactly, and compatibility is satisfied on the average; in system B, compatibility is satisfied exactly, and the stress boundary conditions are satisfied on the average Perturbation and exact integral expressions are found for the frequencies of vibration The response to delta-function, step-function, and harmonic-function loading is examined Dynamic buckling is predicted by shock response method

Nomenclature

A,B = time dependent amplitudes, displacement and stress functions
D = bending stiffness = $Eh^3/12(1 - \nu^2)$
E = Young's modulus
F = stress function
G = energy parameter [Eq (16)]; greater definition of all parameters may be found in Ref 4
K = energy constant [Eq (15)]
L = length of the panel
N_x,N_y,N_{xi},N_{yi} = stress resultants
P = pressure loading
Q = generalized force
R = step function amplitude
R_Δ = response ratio
S = relative displacement of the panel
T = period of vibration
AR = aspect ratio
a = cylinder radius
g = nonlinearity parameter
h = panel thickness
m = mass per unit area
n = circumfeiential mode number
p = ψ_τ
q = reduced nondimensional forcing function
r = reduced step amplitude
t = time variable
w = panel displacement
$x,y\ z$ = space variables
α = reduced amplitude parameter
β = energy parameter [Eq (24)]
γ = frequency parameter [Eq (A7)]
δ = perturbation parameter
Δ = displacement parameter
ϵ = nonlinearity parameter
θ = special form of ψ
χ = nonlinearity parameter
η,μ = nonlinearity parameter
τ = nondimensional time
ϕ = special form of ψ
ψ = nondimensional amplitude
ω = frequency
Ω = nondimensional frequency

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Introduction

TWO solutions to the shallow-shell equations are considered in this paper The first satisfies the stress boundary conditions exactly, but (in accordance with the Galerkin method) it satisfies compatibility only on the average This development is referred to as system A The second solution